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A selfadjoint second order hyperbolic system

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Abstract

We consider a second order hyperbolic system of the type

$$Lu = u_{tt} - Bu_{xx} = f(x,t), \quad (x,t) \in T_m, \tag{1}$$

where matrix B is a nonsingular constant matrix with positive eigenvalues, $(x,t) \in R^2$ and $u, f \in R^n$. The set T_m is defined to be

$$T_m = \{(x,t) \mid 0 \leqslant t \leqslant 1/m,\ |x| \leqslant 1 - mt\}, \tag{2}$$

where $m = \min\{\mu_k\}$ and μ_k^2 is any eigenvalue of the matrix B. We will show that, under the condition $u(x,0) = 0$, $|x| \leqslant 1$, a symmetric Green's function $G_{n\times n}$ can be constructed [K. Kreith, A selfadjoint problem for the wave equation in higher dimensions, Comput. Math. Appl. 21 (5) (1991) 12–132] so that

$$u(x,t) = \iint\limits_{T_m} G_{n\times n}(x,t;\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \tag{3}$$

for any function $f \in L^2(T_m)$. This will imply that the operator L in (1) over the set $L^2(T_m)$ of functions given by Eq. (3) and $u(x,0) = 0$, $|x| \leqslant 1$, is selfadjoint. We also note that the same result holds for u in (1), under the condition that $u_t(x,0) = 0$, $|x| \leqslant 1$. We further note that when B has only one eigenvalue μ^2, the function u in Eq. (3) satisfies a boundary condition similar to that of Kalmenov [T. Kalmenov, On the spectrum of a selfadjoint problem for the wave equation, Akad. Nauk. Kazakh SSR Vestnik 1 (1983) 63–66] on the characteristic boundaries of T_μ.

Keywords: Selfadjoint; Hyperbolic system; Eigenvalue; Characteristic triangle; Fundamental solutions; Jordan canonical form; Laplace transform; Fourier transform

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1. Introduction

In [2] K. Kreith constructed Green's functions G_i, $i = 1, 2, 3$, for the equation

$$Lu = u_{tt} - \Delta u = f(x,t), \quad (x,t) \in T_1, \tag{4}$$

where T_1 is

$$T_1 = \{(x,t) \mid 0 \leqslant t \leqslant 1,\ |x| \leqslant 1 - t\} \tag{5}$$

for the cases where $x \in R^i$, $i = 1, 2, 3$, which gave rise to selfadjointness of the operator in (4) over the set $L^2(T_1)$ of functions given by

$$u(x,t) = \iint\limits_{T_1} G_1(x,t;\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau \tag{6}$$

and satisfying $u(x,0) = 0$, $|x| \leqslant 1$. In addition he showed that in the case $x \in R$ the function u given by (6) satisfies the additional characteristic boundary condition of Kalmenov [1], namely

$$u(x,t) = u\big(\big(1 + (x+t)\big)/2, \big(1 - (x+t)\big)/2\big), \quad (x,t) \in T_1. \tag{7}$$

These are important results because selfadjoint problems arise in the case of elliptic equations. The fact that (4), (5) is selfadjoint implies that there are real eigenvalues and corresponding eigenfunctions associated with the operator L in (4) over the set $L^2(T_1)$. In what follows we extend these results to systems of second order hyperbolic equations in two variables $(x,t) \in R^2$. The selfadjointness result will indicate that the operator L in the hyperbolic system

$$Lu = u_{tt} - Bu_{xx} = f(x,t), \quad (x,t) \in T_m, \tag{8}$$

where

$$T_m = \{(x,t) \mid 0 \leqslant t \leqslant 1/m,\ |x| \leqslant 1 - mt\}, \tag{9}$$

subject to either the condition

$$u(x,0) = 0, \quad |x| \leqslant 1, \tag{10}$$

or the condition

$$u_t(x,0) = 0, \quad |x| \leqslant 1, \tag{11}$$

has an infinite set of real eigenvalues and corresponding set of eigenfunctions in $L^2(T_m)$, where we assume that $\det B \neq 0$. We note here that in [3] the hyperbolicity of a general second order system in two variables is defined as follows. Given $Au_{xx} + 2Bu_{xy} + Cu_{yy} + A_1u_x + B_1u_y + C_1u = F$, with A, B, C, A_1, B_1, C_1 all matrices and $\det C \neq 0$, denote by $p(\xi,\eta) = \det(A\xi^2 + 2B\xi\eta + C\eta^2)$. Then if $p(1,\lambda)$ has only real roots the system is called hyperbolic. In this case the characteristics of the system satisfy $dx + \lambda_i\, dy = 0$, where λ_i are the roots of the characteristic polynomial $p(1,\lambda)$. Accordingly, to guarantee the hyperbolicity of the system (8) we assume that the eigenvalues μ_k^2 of B are all positive and denote $m = \min\{\mu_k\}$. In what follows, we first find a fundamental solution for the system (8) and then construct the symmetric Green's function $G_{n\times n}$ using the technique of Kreith [2], so that u satisfies

$$u(x,t) = \iint\limits_{T_m} G_{n\times n}(x,t;\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau. \tag{12}$$

For the special case that matrix B has only one eigenvalue μ^2, we show that u in Eq. (12) satisfies the condition

$$u(x,t) = u\left(\frac{(1+(x+\mu t))}{2}, \frac{(1-(x+\mu t))}{2}\right) \tag{13}$$

on the characteristic boundaries of T_μ.

2. Fundamental solutions

Define the matrix function $C_{n\times n}$ to be the fundamental solution of the system (8) if it satisfies the equation

$$\begin{aligned} &C_{n\times n_{tt}} - BC_{n\times n_{xx}} = \delta(x)\delta(t)I_{n\times n}, \quad t>0,\\ &C_{n\times n} \equiv 0, \quad t \leqslant 0, \end{aligned} \tag{14}$$

where $C_{n\times n}$ is defined to be $C_{n\times n}(x,t) = H(t)v(x,t)_{n\times n}$, for the step function H,

$$H(t) = 0, \quad t<0, \qquad H(t) = 1, \quad t>0,$$

and $v_{n\times n}$ satisfying the equation

$$\begin{aligned} &v_{n\times n_{tt}} - Bv_{n\times n_{xx}} = 0,\\ &v_{n\times n}(x,0^+) = 0, \qquad v_{n\times n_t}(x,0^+) = \delta(x)I_{n\times n}. \end{aligned} \tag{15}$$

The usual Laplace transform of Eq. (15) results in

$$-B\tilde{v}_{n\times n_{xx}} + s^2\tilde{v}_{n\times n} = \delta(x)I_{n\times n}. \tag{16}$$

This followed by the Fourier transform

$$\hat{\tilde{v}}_{n\times n} = \frac{1}{2\pi}\int_R e^{-i\alpha x}\tilde{v}_{n\times n}(x,s)\,dx \tag{17}$$

yields

$$\alpha^2 B\hat{\tilde{v}}_{n\times n} + s^2\hat{\tilde{v}}_{n\times n} = I_{n\times n}. \tag{18}$$

Noticing that the eigenvalues of B are positive, we can solve for $\hat{\tilde{v}}_{n\times n}$, and then apply the inverse Fourier transform to obtain

$$\tilde{v}_{n\times n} = \frac{1}{2\pi}\int_R \left(\alpha^2 B + s^2 I_{n\times n}\right)^{-1} e^{-i\alpha x} I_{n\times n}\,d\alpha. \tag{19}$$

We rewrite the integral in (19) as follows:

$$\tilde{v}_{n\times n} = \frac{1}{2\pi}\int_R \left(\alpha^2 I_{n\times n} + s^2 B^{-1}\right)^{-1} B^{-1} e^{-i\alpha x} I_{n\times n}\,d\alpha. \tag{20}$$

Let J be the Jordan canonical form of the matrix B^{-1} so that for some matrix Q we have $Q^{-1}B^{-1}Q = J$. After this substitution for B^{-1} and some rearrangement of the terms we have

$$\tilde{v}_{n\times n} = Q\left(\frac{1}{2\pi}\int_R \left(\alpha^2 I_{n\times n} + s^2 J\right)^{-1} e^{-i\alpha x} I_{n\times n}\,d\alpha\right) J Q^{-1}. \tag{21}$$

We observe here that an A_k block of $\alpha^2 I_{n\times n} + s^2 J$ corresponding to a typical J_k block of J is either a 1×1 matrix consisting of $\alpha^2 + s^2\lambda_k$ or is an $n_k \times n_k$ sub matrix of J with $\alpha^2 + s^2\lambda_k$ along its main diagonal and s^2 along super diagonal. The matrix $(\alpha^2 I_{n\times n} + s^2 J)^{-1}$ then, has blocks that are either 1×1 matrices consisting of $\frac{1}{\alpha^2+s^2\lambda_k}$ or $n_k \times n_k$ sub matrices of the form

$$A_k^{-1} = \begin{pmatrix} \frac{1}{\alpha^2+s^2\lambda_k} & \frac{-s^2}{(\alpha^2+s^2\lambda_k)^2} & \frac{(s^2)^2}{(\alpha^2+s^2\lambda_k)^3} & \cdots & \frac{(-s^2)^{n_k-1}}{(\alpha^2+s^2\lambda_k)^{n_k}} \\ 0 & \frac{1}{\alpha^2+s^2\lambda_k} & \frac{-s^2}{(\alpha^2+s^2\lambda_k)^2} & \cdots & \frac{(-s^2)^{n_k-2}}{(\alpha^2+s^2\lambda_k)^{n_k-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\alpha^2+s^2\lambda_k} \end{pmatrix}.$$

We now evaluate the integral in Eq. (21) using the fact that the entries of the matrix $(\alpha^2 I_{n\times n} + s^2 J)^{-1} e^{-i\alpha x} I_{n\times n}$ are made up of the entries such as those in the matrix A_k^{-1}. Using the Cauchy Principal value of the integrals involved [4], we will have

$$\frac{1}{2\pi}\int_R \frac{1}{(\alpha^2 + s^2\lambda_k)} e^{-i\alpha x}\, d\alpha = \frac{e^{-\sqrt{\lambda_k}|x|}}{2\sqrt{\lambda_k}s}, \tag{22}$$

$$\begin{aligned} &\frac{1}{2\pi}\int_R \frac{(-s^2)^m}{(\alpha^2 + s^2\lambda_k)^{m+1}} e^{-i\alpha x}\, d\alpha \\ &\quad = \left(s^2\right)^m \left(\frac{(-1)^m (m+1)! e^{-\sqrt{\lambda_k}s|x|}}{(2\sqrt{\lambda_k}s)^{2m+1}} + \frac{(-|x|)^m e^{-\sqrt{\lambda_k}s|x|}}{(2\sqrt{\lambda_k}s)^{m+1}} \right. \\ &\qquad \left. + \sum_{l=1}^{m-1} \frac{(m+1)!}{(m-l+1)!} \binom{m}{l} \frac{(-1)^l (|x|)^{m-l} e^{-\sqrt{\lambda_k}s|x|}}{(2\sqrt{\lambda_k}s)^{m+l+1}} \right), \quad m \geqslant 1. \end{aligned} \tag{23}$$

Using Eqs. (22) and (23) we convert $\tilde{v}_{n\times n}$ to $v_{n\times n}$ by the inverse Laplace transform. For simplicity, we only write the inverse Laplace transforms of the entries along the main diagonal of $(\alpha^2 I_{n\times n} + s^2 J)^{-1}$ and the inverse Laplace transform of a typical off diagonal entry.

$$L^{-1}\left(\frac{e^{-\sqrt{\lambda_k}|x|}}{2\sqrt{\lambda_k}s}\right) = \frac{1}{2\sqrt{\lambda_k}} H\left(t - \sqrt{\lambda_k}|x|\right), \tag{24}$$

$$\begin{aligned} &L^{-1}\left(\left(s^2\right)^m \left(\frac{(-1)^m (m+1)! e^{-\sqrt{\lambda_k}s|x|}}{(2\sqrt{\lambda_k}s)^{2m+1}} + \frac{(-|x|)^m e^{-\sqrt{\lambda_k}s|x|}}{(2\sqrt{\lambda_k}s)^{m+1}} \right.\right. \\ &\qquad \left.\left. + \sum_{l=1}^{m-1} \frac{(m+1)!}{(m-l+1)!} \binom{m}{l} \frac{(-1)^l (|x|)^{m-l} e^{-\sqrt{\lambda_k}s|x|}}{(2\sqrt{\lambda_k}s)^{m+l+1}} \right)\right) \\ &\quad = \frac{(-1)^m (m+1)!}{2^{2m+1}(\sqrt{\lambda_k})^{2m+1}} H\left(t - \sqrt{\lambda_k}|x|\right) + \frac{(-1)^{2m-1}|x|^m}{(2)^{m+1}(\sqrt{\lambda_k})^{2m}} \left(\frac{d}{d|x|}\right)^{m-1} \delta\left(t - \sqrt{\lambda_k}|x|\right) \\ &\qquad + \sum_{l=1}^{m-1} \frac{(m+1)!}{(m-l+1)!} \binom{m}{l} \frac{(-1)^{m-1}|x|^{m-l}}{2^{m+l+1}\lambda_k^m} \left(\frac{d}{d|x|}\right)^{m-l-1} \delta\left(t - \sqrt{\lambda_k}|x|\right), \quad m \geqslant 1. \end{aligned} \tag{25}$$

If we denote

$$L^{-1}\left(\frac{1}{2\pi}\int_R \left(\alpha^2 I_{n\times n}+s^2 J\right)^{-1} e^{-i\alpha x} I_{n\times n}\,d\alpha\right)=E_{n\times n}(x,t), \tag{26}$$

where $E_{n\times n}$ is the matrix with elements described as in (24)–(25), then the fundamental solution $C_{n\times n}$ for Eq. (8) will be

$$C_{n\times n}(x,t;\xi,\tau)=H(t-\tau)QE_{n\times n}(x-\xi,t-\tau)JQ^{-1}. \tag{27}$$

3. Symmetric Green's function

Now we construct our symmetric Green's function $\tilde{G}_{n\times n}$ for the operator in Eq. (8) by the same technique as in [2]. First we note that $\tilde{G}_{n\times n}$ is of the form

$$\tilde{G}_{n\times n}(x,t)=H(t)u_{n\times n}(x,t)+H(-t)v_{n\times n}(x,t), \tag{28}$$

where $u_{n\times n}$ and $v_{n\times n}$ satisfy

$$\begin{aligned} &u_{n\times n_{tt}}-Bu_{n\times n_{xx}}=0, \quad t>0,\\ &u_{n\times n}(x,0^+)=0, \qquad u_{n\times n_t}(x,0^+)=\frac{1}{2}\delta(x)I_{n\times n}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} &v_{n\times n_{tt}}-Bv_{n\times n_{xx}}=0, \quad t<0,\\ &v_{n\times n}(x,0^+)=0, \qquad v_{n\times n_t}(x,0^+)=-\frac{1}{2}\delta(x)I_{n\times n}. \end{aligned} \tag{30}$$

Then, not only is $\tilde{G}_{n\times n}$ a fundamental solution of Eq. (8) by a proof similar to the one in [2], but also symmetric in the sense that

$$\tilde{G}_{n\times n}(x,t;\xi,\tau)=\tilde{G}_{n\times n}(\xi,\tau;x,t). \tag{31}$$

The resulting $\tilde{G}(x,t;\xi,\tau)$ is then

$$\tilde{G}_{n\times n}(x,t;\xi,\tau)=\frac{1}{2}C_{n\times n}(x,t;\xi,\tau)+\frac{1}{2}C_{n\times n}(x,-t;\xi,-\tau), \tag{32}$$

where $C_{n\times n}$ is given by Eq. (27). If we further amend $\tilde{G}_{n\times n}$ to

$$G_{n\times n}(x,t;\xi,\tau)=\tilde{G}_{n\times n}(x,t;\xi,\tau)-\tilde{G}_{n\times n}(x,-t;\xi,\tau). \tag{33}$$

Then $G_{n\times n}$ is still symmetric and in addition it satisfies $G_{n\times n}(x,0;\xi,\tau)\equiv 0$, $|x|\leqslant 1$. This, in turn provides the following solution to (8), (10), namely,

$$u(x,t)=\iint_{T_m} G_{n\times n}(x,t;\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau. \tag{34}$$

On the other hand, one can observe that if we introduce

$$g_{n\times n}(x,t;\xi,\tau)=\tilde{G}_{n\times n}(x,t;\xi,\tau)+\tilde{G}_{n\times n}(x,-t;\xi,\tau), \tag{35}$$

then, $g_{n\times n_t}(x,0;\xi,\tau)\equiv 0$, $|x|\leqslant 1$. And therefore

$$u(x,t)=\iint_{T_m} g_{n\times n}(x,t;\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau \tag{36}$$

satisfies Eqs. (8), (11). Based on these observations, we have the following.

Theorem 1. *Let f be a function in the set $L^2(T_m)$. Then the operator L in Eq.* (8) *defined on the set $L^2(T_m)$ of functions given by* (34) *or* (36) *and satisfying $u(x,0)=0$, $|x|\leqslant 1$, or $u_t(x,0)=0$, $|x|\leqslant 1$, respectively, is selfadjoint.*

4. The characteristic triangle

In order to investigate the similarity of our result with those of Kreith [2] and Kalmenov [1] consider the Eq. (8) with B having only one eigenvalue μ^2, and holding in the characteristic triangle T_μ, i.e.,

$$\begin{aligned} &Lu = u_{tt} - Bu_{xx} = f(x,t), \quad (x,t)\in T_\mu, \\ &T_\mu = \bigl\{(x,t) \mid 0\leqslant t\leqslant 1/\mu,\ -1+\mu t\leqslant x\leqslant 1-\mu t\bigr\}. \end{aligned} \tag{37}$$

Based on our calculations in Section 2, $E_{n\times n}$ in Eq. (26) satisfies $E(x,t;\xi,\tau)_{n\times n}=0$ for $t-\tau-\mu|x-\xi|<0$ and

$$E_{n\times n}(x,t;\xi,\tau) = \begin{pmatrix} \frac{1}{2\mu} & \frac{-2}{2^3\mu^3} & \frac{3!}{2^5\mu^5} & \cdots & \frac{(-1)^{n-1}n!}{2^{2n-1}\mu^{2n-1}} \\ 0 & \frac{1}{2\mu} & \frac{-2}{2^3\mu^3} & \cdots & \frac{(-1)^{n-2}(n-1)!}{2^{2n-3}\mu^{2n-3}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2\mu} \end{pmatrix}, \quad t-\tau-\mu|x-\xi|>0.$$

Let A be

$$A = \frac{-1}{2}Q\begin{pmatrix} \frac{1}{2\mu} & \frac{-2}{2^3\mu^3} & \frac{3!}{2^5\mu^5} & \cdots & \frac{(-1)^{n-1}n!}{2^{2n-1}\mu^{2n-1}} \\ 0 & \frac{1}{2\mu} & \frac{-2}{2^3\mu^3} & \cdots & \frac{(-1)^{n-2}(n-1)!}{2^{2n-3}\mu^{2n-3}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2\mu} \end{pmatrix} JQ^{-1}.$$

Then, the Green's function $G_{n\times n}(x,t;\xi,\tau)$ in T_μ will have values as shown in the characteristic triangle T_μ in Fig. 1.

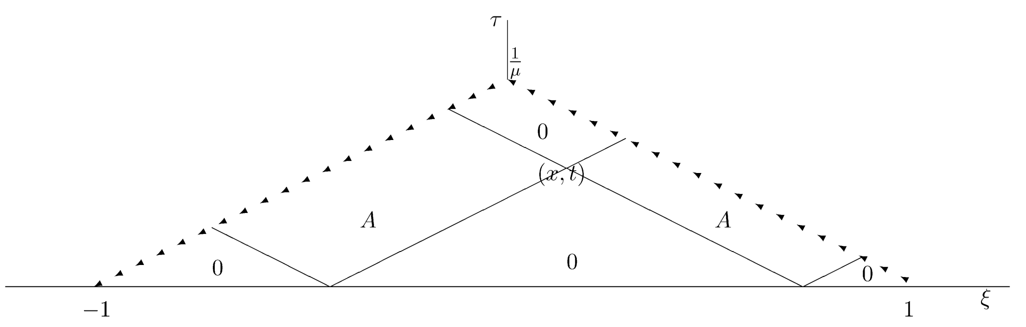

Fig. 1. $G_{n\times n}$ in the characteristic triangle.

From Fig. 1, we observe that the solution u satisfies the condition

$$u(x,t) = u\left(\frac{(1+(x+\mu t))}{2}, \frac{(1-(x+\mu t))}{2}\right) \tag{38}$$

on the characteristic boundaries of T_μ.

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